Introduction to Smooth Manifolds, by John Lee
Chapter 9 Solutions

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Problem 9-1. Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.

(a) We say γ is **periodic** if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and nonconstant.

(b) Show that if γ is periodic and nonconstant, then there exists a unique positive number T (called the **period of** γ) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}, S^1$, or $\mathbb{R}^0$.

Solution. (a) Let the domain of γ be (a, b). Assume it is not constant nor injective. Let $t < t'$ be with $\gamma(t) = \gamma(t')$. Let $T = t' - t > 0$. Then Theorem 9.12b tells us that

$$\mathcal{D}^{\gamma(t+T)} = \mathcal{D}^{\gamma(t)} - T.$$

However, $\mathcal{D}^{\gamma(t)}$ is the same as $\mathcal{D}^{\gamma(t+T)}$, so $\mathcal{D}^{\gamma(t)}$ must be $\mathbb{R}$. On the other hand, letting θ be the flow induced from X,

$$\theta(s+T, \gamma(t)) = \theta(s, \gamma(t+T)) = \theta(s, \gamma(t)),$$

for all real s. This shows $\theta^{(\gamma(t))} = \gamma$ is periodic (and nonconstant, as we wanted).

(b) There is T, the smallest positive number such that $\gamma(t+T) = \gamma(t)$ for all t.

If it doesn't exist, then there is an infimum T^* of those periods, and hence a Cauchy sequence that converges to it. It is easy to see that if T_1 and T_2 are in the set of periods, then so does $|T_1 - T_2|$. Since the sequence is Cauchy we conclude the infimum T^* must be zero.

Fix some valid time t_0. Then $\gamma(t_0)$ is in every neighborhood of $\gamma(t)$ in M, for every t, by choosing appropriately small period $\varepsilon > 0$ for the size of the neighborhood. That is, we choose a neighborhood of $\gamma(t)$ in M, which by continuity contains $\gamma(t-\varepsilon, t+\varepsilon)$ for some $\varepsilon > 0$. Then we can choose a period p small enough such that t_0 plus a multiple of p lies in $(t-\varepsilon, t+\varepsilon)$.

This conclusion contradicts the Hausdorff property of M, so we conclude there is a smallest period.

Now it is easy to see that all other periods are just multiples of T, else we can subtract T enough times to find a yet smaller period.

In particular, by the argument of (a) we know $\gamma(t) = \gamma(t')$ if and only if $t - t'$ is in the set of periods for γ. That is, T satisfies the property of the **period**.

To show unicity, just consider that another period kT, for integer k would not divide $(t+T) - t$, even though $\gamma(t) = \gamma(t+T)$.

(c) If γ is constant, then the image of γ is diffeomorphic to $\mathbb{R}^0$.

If γ is nonconstant, its derivative never vanishes, by Prop. 9.12. This means γ is a smooth immersion of the open interval J into M. If γ is injective, this means it is an injective smooth immersion, and J (which is diffeomorphic to $\mathbb{R}$) is diffeomorphic to the image of γ by Prop. 5.18.

If γ is periodic and nonconstant, we can consider the smooth covering map

$$\pi : \mathbb{R} \to S^1, \quad t \mapsto e^{2\pi i t/T},$$

where T is the period as in (b). Since γ is constant on the fibers of π, we must have a map $\tilde{\gamma} : S^1 \to M$ with $\tilde{\gamma} \circ \pi = \gamma$. Since $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$, $\tilde{\gamma}$ is injective.

Now since π is a local diffeomorphism, $\tilde{\gamma}$ is also a smooth immersion. By Prop. 5.18, the image of γ is diffeomorphic to S^1, as we wanted to show.

□

Problem 9-2. Suppose M is a smooth manifold, $S \subseteq M$ is an immersed submanifold, and V is a smooth vector field on M that is tangent to S.

(a) Show that for any integral curve γ of V such that $\gamma(t_0) \in S$, there exists $\varepsilon > 0$ such that $\gamma((t_0 - \varepsilon, t_0 + \varepsilon)) \subseteq S$.

(b) Now assume S is properly embedded. Show that any integral curve that intersects S is contained in S.

(c) Give a counterexample to (b) if S is not closed.

Solution. (a) Consider $i : S \hookrightarrow M$ the smooth immersion. There's V tangent to S, and there is U, a vector field in S that is i-related to V by Prop 8.23. Given the integral curve γ of V through $\gamma(t_0)$, there is also an integral curve of U through $i^{-1}(\gamma(t_0))$, which we call η. We choose a parametrization of η so that $i \circ \eta(t_0) = \gamma(t_0)$. By Prop. 9.2, η is defined on a neighborhood of t_0.

By Prop. 9.6, $i \circ \eta$ is an integral curve of V passing through $\gamma(t_0)$, and by unicity of integral curves $i \circ \eta = \gamma$ in some neighborhood of t_0. In particular this shows there is $\varepsilon > 0$ with $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) = i \circ \eta(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq S$.

(b) Assume that γ starts at $p \in S$. Assume by way of contradiction that γ is not contained in S. Without loss of generality assume that there is a positive time where γ is outside S (the negative case is handled similarly). Let t^* be the infimum of $t > 0$ such that $\gamma(t) \notin S$. By part (b) we must have that this infimum is in the set itself, else $\gamma(t^*) \in S$ and we can find a larger lower bound. This means that $\gamma(t^*/2, t^*) \subseteq S$. Since S is closed,

$$\gamma(t^*) \in \gamma(\overline{(t^*/2, t^*)}) \subseteq \overline{\gamma(t^*/2, t^*)} \subseteq S,$$

a contradiction. We conclude the infimum doesn't exist and the image of γ is contained in S.

(c) Consider the open unit disk in $\mathbb{R}^2$, and the Euler vector field. Any radial maximal integral curve intersects the disk, but is not contained in it.

□

Problem 9-4. For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $S^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t,z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth nonvanishing vector field on S^{2n-1}. [Remark: case $n = 2$, the integral curves of X are the curves γ_z of Problem 3-6, so this provides a simpler proof that each γ_z is smooth.]

Solution. If the flow is θ we have $\dot{\theta}^{(z)}(0) = ie^{it}z(0) = iz \neq 0$. This shows the infinitesimal generator of θ is a smooth nonvanishing vector field on S^{2n-1}. $\square$

Problem 9-5. Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.

Solution. Since M is compact, the nowhere vanishing smooth vector field V generates a complete flow. This also gives that each $\theta_t : M \to M_t = M$ is a diffeomorphism homotopic to the identity with homotopy θ:

The map will be injective because $\theta_t(p) = \theta_t(q)$ implies $p = q$. It will be surjective because $\theta_{-t}(p)$ maps to p. And it is a local diffeomorphism because

$$\theta_t \circ \theta_{-t} = \mathrm{Id}$$
$$\theta_{-t} \circ \theta_t = \mathrm{Id}\,.$$

Since all points are regular points for V, we cover M with regular coordinate cubes $(s^1, ..., s^n)$ where in each, $V_p = \left.\dfrac{\partial}{\partial s^1}\right|_p$. Then for each cube we consider even smaller (of half side) regular coordinate cubes so that in each we can move some t_i time forward in the flow and not have any fixed points (that is, $\theta_{t_i}|_{C_i}$ does not have fixed points, where C_i are the smaller cubes).

By compactness of M, we can choose finitely many of these cubes C_i to cover M. Let $t = \min t_i$, and

$$\theta_t : M \to M$$

will be by construction a smooth map homotopic to the identity θ_0 with no fixed points.

Obs: we can say more, that it will be a smooth isotopy of M with no fixed points. $\square$

Problem 9-6. Prove the escape lemma. Suppose M is a smooth manifold and $V \in \mathfrak{X}(M)$. If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.

Solution. Suppose there is t_0 such that $\gamma[t_0, b)$ is precompact. Then let K be its closure. One can now cover K with precompact coordinate balls. Then one can also consider the same neighborhoods but with half the radii. These are also pre-compact. Since these also cover K, we can take finitely many, and call them $U_1, ..., U_m$.

Now we consider a bump function $\psi : M \to \mathbb{R}$ such that it is one on $\bigcup_1^m \overline{U_j}$, and supported on $\bigcup_1^m U_j^2$, where U_j^2 are the original precompact neighborhoods.

In this construction ψV is now a compactly supported smooth vector field of M, thus generating a complete flow.

Since the two vector fields agree on $\bigcup_1^m U_j$, we conclude they generate the same flow there. In particular, in $\bigcup_1^m U_j$, V and ψV should generate the same integral curve through $\gamma(t_0)$ (with translations to make sure that $\tilde{\gamma}(t_0) = \gamma(t_0)$). We can consider now $\tilde{\gamma}(b)$, which will be in

$$\tilde{\gamma}\overline{(t_0, b)} \subseteq \overline{\gamma(t_0, b)} \subset K \subset \cup_i^m U_j,$$

so that $\tilde{\gamma}$ extends γ further. This contradicts the maximality of γ. We conclude $\gamma[t_0, b)$ is not contained in any compact subset of M.

□

Problem 8-4. Let M be a smooth manifold with boundary. Show that there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

Solution. We show it for inward-pointing, and the outward pointing is achieved by multiplying the field by -1.

We cover the boundary of M with regular coordinate half-balls, and then consider the same regular coordinate half-balls with half the radius.

In each of the larger regular coordinate half-balls we can define a smooth vector field by $V_i = \dfrac{\partial}{\partial x^n}$. Then in each, we consider the bump function which is 1 in the closure of the smaller half-ball, and vanishes outside the larger half-ball. If we denote the half ball by U_i and the larger by U_i^2, we have a bump function $b_i : M \to \mathbb{R}$ which is 1 on U_i and 0 outside U_i^2.

Then $b_i V_i$ becomes a smooth vector field on M which is inward pointing in $\partial M \cap U_i$.

We now consider the open cover of M consisting of U_i and $M \setminus \partial M$, and a partition of unity subordinate to it, denoted $\psi_i : M \to \mathbb{R}$.

We claim $V = \sum \psi_i b_i V_i$ is everywhere inward-pointing.

For every $p \in \partial M$, the vector field becomes $\sum_{i=1}^k \psi_i V_i$, where $U_1, ..., U_k$ are the patches that contain p. Since each $(V_i)_p$ is inward pointing, we have a sum of inward pointing (or zero) tangent vectors, which must then be inward-pointing also.

□

Problem 9-7. Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. [Hint: first prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.]

Solution. Let p and q be points in the unit ball in $\mathbb{R}^n$. Take coordinates for it such that $q - p$ is parallel to the x^n-axis. Let $V = q - p = |q - p| \dfrac{\partial}{\partial x^n}$ for every point of the ball. Then it is clear this will generate a complete flow in the ball, and that $\theta_1(p) = q$. To make it compactly supported, consider the bump function $b : \mathbb{R}^n \to \mathbb{R}$ that is 1 on the smallest closed ball containing both p and q (so radius $\max |p|, |q|$) and vanishes outside $\mathbb{B}^n$. Then bV is the vector field we're looking for.

For the smooth connected manifold M, we take a smooth path $\gamma : I \to M$ from p to q, and cover it with finitely many regular coordinate balls whose center lies in the path, by compactness of I. Then we can order these by their centers, and call them $U_1, ..., U_k$. We must have that U_i and U_{i+1} intersect, so we can take points in the intersection of adjacent coordinate balls, so $p = p_0 \in U_1$, $p_1 \in U_1 \cap U_2$, ..., $p_k = q \in U_k$. From the construction on $\mathbb{B}^n$ above we can build V_i, smooth vector fields in M, compactly supported in U_i, such that the flow θ_i they generate is complete and satisfy $(\theta_i)_1(p_{i-1}) = p_i$.

The composition
$$(\theta_k)_1 \circ (\theta_{k-1})_1 \circ \cdots \circ (\theta_1)_1 : M \to M$$
is a diffeomorphism, and it brings p to q, as we wanted. $\square$

Problem 9-11. Prove Theorem 9.24:

(Boundary Flowout Theorem) Let M be a smooth manifold with nonempty boundary, and let N be a smooth vector field on M that is inward-pointing at each point of ∂M. There exist a smooth function $\delta : \partial M \to \mathbb{R}^+$ and a smooth embedding $\Phi : \mathcal{P}_\delta \to M$, where

$$\mathcal{P}_\delta = \{(t,p) : p \in \partial M, 0 \leq t < \delta(p)\} \subseteq \mathbb{R} \times \partial M,$$

such that $\Phi(\mathcal{P}_\delta)$ is a neighborhood of ∂M, and for each $p \in \partial M$ the map $t \mapsto \Phi(t,p)$ is an integral curve of N starting at p.

Solution. Since N is smooth on ∂M, at every p there is a regular coordinate chart $\varphi : U \to \mathbb{H}^n$ such that $N \circ \varphi^{-1}$ has an extension to a smooth vector field on a neighborhood of $\mathbb{H}^n$.

Since $N_p \neq 0$, we can choose the chart and the neighborhood such that $\hat{N}$ has only regular points in this neighborhood of $\mathbb{H}^n$.

By Theorem 9.20, considering the embedded hypersurface

$$\{x^n = 0\} = \varphi(U \cap \partial M) = S,$$

and the flow $\theta : \mathcal{D} \to \mathbb{R}^n$ of $\hat{N}$ which has $\hat{N}_p \notin T_pS$, we have a smooth positive function $\delta : \varphi(U \cap \partial M) \to \mathbb{R}^+$ and a domain $\mathcal{O}_\delta \subseteq \mathcal{O} \subseteq \mathcal{D} \cap (\mathbb{R} \times S)$ such that $\Phi = \theta|_{\mathcal{O}_\delta}$ is a diffeomorphism onto an open submanifold of $\mathbb{R}^n$ (containing S).

If we consider instead $\mathcal{P}_\delta = \{(t,p) : p \in S, 0 \leq t < \delta(p)\} \subseteq \mathcal{O}_\delta$, we can show that $\Phi(\mathcal{P}_\delta) \subseteq \mathbb{H}^n$.

In fact, if for some $(t,q) \in \mathcal{P}_\delta$, $\Phi(t,q)$ is in the lower half-space, then the integral curve $\theta^{(q)} : [0, \delta(p)) \to \mathbb{R}^n$, which starts at q with upward-pointing tangent vector (in our coordinates an inward-pointing vector in $T_q\partial M$ corresponds to an upward-pointing vector in $T_{\varphi(q)}\mathbb{R}^n$) would have to return to the boundary. At the first such point N would have to be tangent to ∂M or outward-pointing.

In addition, the paragraph above indicates that no point in the image of $\mathcal{O}_\delta \smallsetminus \mathcal{P}_\delta$ is in $\mathbb{H}^n$. A negative time t with $\theta(t,q)$ in $\{x^n \geq 0\}$ would have to return to the boundary $\{x^n = 0\}$ before or at time 0, which would imply that N is outward-pointing or tangent to ∂M at the first such point.

This shows that Φ sends $\mathcal{P}_\delta$ diffeomorphically to $\Phi(\mathcal{O}_\delta) \cap \mathbb{H}^n$, which is an open submanifold of $\mathbb{H}^n$ containing its boundary (since $\Phi(\mathcal{O}_\delta)$ is open in $\mathbb{R}^n$ containing $\{x^n = 0\}$).

Sending it back to M, we showed the existence of a $\widetilde{\mathcal{P}_\delta} = (\mathrm{Id} \times \varphi^{-1})(\mathcal{P}_\delta) \subseteq \mathbb{R} \times (U \cap \partial M)$ associated to a smooth function $\delta \circ \varphi : U \cap \partial M \to \mathbb{R}^+$, a smooth embedding $\varphi^{-1} \circ \Phi \circ (\mathrm{Id} \times \varphi) : \widetilde{\mathcal{P}_\delta} \to M$ such that its image is a neighborhood of $U \cap \partial M$, and we showed also that for each $p \in U \cap \partial M$ the map $t \mapsto \varphi^{-1} \circ \Phi \circ (\mathrm{Id} \times \varphi)(t,p)$ is an integral curve of N starting at p (because $t \mapsto \Phi(t,p)$ is an integral curve of $\hat{N}$ in $\mathbb{H}^n$ starting at p).

We can do this process for every point in the boundary of M, and find an open cover U_i of ∂M with the respective diffeomorphisms $\Phi_i : \mathcal{P}_{\delta_i} \to U_i$ and smooth functions $\delta_i : U_i \cap \partial M \to \mathbb{R}^+$.

If a point of the boundary p lies in the intersection of U_i and U_j, then by uniqueness of the integral curve of N starting at p (when it exists), $\Phi_i(t,p)$ and $\Phi_j(t,p)$ agree on their common domain.

This shows we can glue the Φ_i together into a single map Φ on $\bigcup_i \mathcal{P}_{\delta_i} \subseteq \mathbb{R} \times \partial M$, containing $\{0\} \times \partial M$.

1) It will automatically satisfy that $t \mapsto \Phi(t,p)$ is an integral curve starting at p.

This map will have a bijective differential at each point, inherited from the Φ_i.

We claim it will also inherit the injectivity.

If $\Phi_i(t,p) = \Phi_j(t',p')$ for $(t,p) \neq (t',p')$, then we'd be having two integral curves of N, one starting at p and the other at p', which meet in a single point in $U_i \cap U_j$ (which cannot be in the boundary, lest $t = t' = 0$, $p = p'$ since these curves do not return to the boundary).

This implies these two curves agree on all of their image on $\mathrm{Int}M$ (up to translation of domain by $t' - t$, assuming $t < t'$) by the uniqueness of integral curves on a manifold without boundary. At this point we are considering these two curves being extended to the largest domain possible in $\mathrm{Int}M$.

This scenario cannot happen because if we look at the curve around p in U_i coordinates, it converges to p. This would mean in coordinates

$$p = \lim_{s \to 0} \Phi_i(s,p) = \lim_{s \to t'-t} \Phi_j(s,p') = \Phi_j(t'-t,p')$$

and again we would have $t' - t = 0$.

2) We conclude Φ is an injective smooth immersion of an n-dimensional manifold, and must thus be a smooth embedding onto a neighborhood of ∂M.

We now just need to show that $\bigcup_i \mathcal{P}_{\delta_i}$ has an open subset of the form

$$\{(t,p) : p \in \partial M, 0 \leq t < \delta(p)\} \quad (*)$$

for some positive smooth function $\delta : \partial M \to \mathbb{R}^+$. The restriction of Φ to such set will also be a smooth embedding, and will automatically satisfy all the conditions of the theorem.

Consider a partition of unity subordinate to the open cover U_i, which we denote ψ_i.

Now we consider $\delta(p) = \sum \psi_i(p)\delta_i(p)$. This will clearly be positive and smooth.

It is evident that $\delta \leq \max_i \delta_i$ so that $\mathcal{P}_\delta \subseteq \bigcup_i \mathcal{P}_{\delta_i}$ is an open submanifold containg $\{0\} \times \partial M$, as we wanted.

□